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CBR Temperature Fluctuations Induced by Gravitational Waves in a Spatially-Closed Inflationary Universe

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Abstract

Primordial gravitational waves are created during the de Sitter phase of an exponentially-expanding (inflationary) universe, due to quantum zero-point vacuum fluctuations. These waves produce fluctuations in the temperature of the Cosmic Background Radiation (CBR). We calculate the multipole moments of the correlation function for these temperature fluctuations in a spatially-closed Friedman-Robertson-Walker (FRW) cosmological model. The results are compared to the corresponding multipoles in the spatially-flat case. The differences are small unless the density parameter today, Ω_0, is greater than 2.

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I. INTRODUCTION

Inflationary models of the early universe contain a well-studied mechanism which creates primordial fluctuations. The fluctuations originate as quantum-mechanical zero-point fluctuations during the exponentially-expanding de Sitter phase. By a process which may be variously described as particle (graviton) production, non-adiabatic amplification, or super-radiant scattering, these fluctuations become large in the present epoch. As the universe expands, these perturbations are redshifted to longer wavelengths and amplified; during the present epoch these perturbations typically persist over a range of wavelengths λ from 10^{-27} cm $< \lambda < 10^{2}$ cm. For a review of perturbations in inflationary models, see Kolb and Turner [1].

The perturbations of the gravitational field may be decomposed into scalar, vector and tensor components. The tensor perturbations considered in this paper may be thought of as gravitational waves in a classical description, or as spin-two gravitons in the quantum mechanical description used in the present work. The modes of interest have present-day frequencies in the range from 10^{-17} Hz to 10^{-12} Hz and have extremely large occupation numbers. Hence they may also be thought of as classical gravitational waves - the two descriptions coincide. The gravitons are created during the de Sitter phase of rapid expansion by the mechanism originally proposed by Parker; the same mechanism creates particles near a black hole or in any other region where the spacetime curvature is large and particle creation is not forbidden by global symmetries or conservation laws. A simple calculation showing how a potentially-observable spectrum of gravitons is created in inflation is given by Allen [2].

The tensor perturbations of the gravitational field produce temperature fluctuations in the CBR, via the Sachs-Wolfe effect. The expected values of the resulting temperature fluctuations are described by the angular correlation function

$$C(\gamma) \equiv \left\langle 0 \left| \frac{\delta T}{T}(\Omega) \frac{\delta T}{T}(\Omega') \right| 0 \right\rangle = \sum_{l=1}^{\infty} \frac{(2l+1)}{4\pi} \langle a_l^2 \rangle P_l(\cos\gamma). \tag{1.1}$$

Here $\delta T/T(\Omega)$ is the fractional temperature fluctuation in the CBR at point Ω on the observer's celestial sphere, γ is the angle between Ω and Ω', and the quantum expectation value is evaluated in the initial state of the universe. The multipole moments $\langle a_l^2 \rangle$ are generally used to describe $C(\gamma)$.

In a recent paper [3] the expected multipole moments $\langle a_l^2 \rangle$ due to tensor perturbations are calculated in a spatially flat $k = 0$ FRW inflationary model. That paper contains a detailed review of previous work on this problem, a comprehensive description of the physical motivation, and a detailed and self-contained "first-principles" calculation. The present work repeats that calculation in the spatially-closed ($k = +1$) case. The only previous work on tensor perturbations in the spatially-closed case is that of Abbott and Schaefer [4]. Note that the angle brackets around a_l^2 serve as a reminder that we are calculating the *expected* or *expectation values* of these multipole moments, not necessarily the values that they might have in any given realization of the universe.

The calculation in this paper follows the previous work by Allen and Koranda [3] very closely. In the present work, we will assume that the reader is familiar with that earlier paper, and present only the bare minimum of detail required to generalize the work to the $k = +1$ case. In Section II we present the $k = +1$ cosmological model and Sachs-Wolfe effect. Section III gives the form of the metric perturbation operator for linearized gravitational fluctuations. Section IV combines these results to obtain an analytic form for the multipole moments $\langle a_l^2 \rangle$. Section V details the method by which these multipole moments were evaluated numerically, and Section VI outlines the results and conclusions of that numerical study.

Throughout this paper, we use units where the speed of light $c = 1$. However for clarity we have retained Newton's gravitational constant G and Planck's constant $\hbar$ explicitly. We choose function branches so that $\sqrt{x} \geq 0$ and $\arcsin(x) \in [-\pi/2, \pi/2]$.

II. THE BACKGROUND SPACE-TIME AND THE SACHS-WOLFE EFFECT

The spacetime considered here has the topology $R \times S^3$ of the static Einstein cylinder, and is covered by coordinates $x^0 = t, x^1 = \chi, x^2 = \theta, x^3 = \phi$ with the ranges $\chi, \theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$. The time coordinate t ranges over a connected open subset of the real line, which we will specify below. The spatial coordinates cover a three-sphere of radius $a(t)$; we refer to this function as the cosmological scale factor. The metric of the spacetime is given by

$$ds^2 = a^2(t)\Big(-dt^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) + h_{ij}(t, \chi, \theta, \phi)dx^i dx^j\Big). \tag{2.1}$$

The metric perturbation h_{ij} is assumed to be small; in its absence the spacetime metric is that of a homogeneous and isotropic $k = +1$ FRW model. With our choice of gauge for the tensor metric perturbations, the indices $i, j = 1, 2, 3$ run only over the spatial coordinates.

In order to completely specify the cosmological model, we need to define the cosmological scale-factor $a(t)$. The cosmological model is completely defined by the free parameters given in Table I. Note that we have assumed that the universe is currently expanding, since we require H_0 to be positive. The density parameter

$$\Omega_0 = \frac{8\pi G \rho_0}{3H_0^2} \tag{2.2}$$

is the ratio of the present-day energy-density ρ_0 to the critical energy density required to produce a spatially-flat $k = 0$ universe.

A. The Matter-Dominated (Dust) Phase

In our cosmological model, the universe is assumed to pass through three "phases", appropriate to a simple inflationary model. We let $t = 0$ denote the present time. The most recent phase was a matter-dominated period of expansion, described by the scale factor

$$a(t) = A\sin^2(t/2 + B) \qquad \text{for} \quad t_{eq} < t < 0. \tag{2.3}$$

Here the constants A, B, t_{eq} are defined by

$$
\begin{aligned}
A &= \frac{1}{H_0}\Omega_0(\Omega_0-1)^{-3/2},\\
B &= \arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}}, \quad \text{and} \\
t_{eq} &= 2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0(1+Z_{eq})}} - 2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}}.
\end{aligned}
\tag{2.4}
$$

During this matter-dominated phase, the stress-energy tensor is that of a perfect fluid, with zero pressure and an energy density proportional to $a^{-3}(t)$. We assume (as indicated in Table I) that the surface of last scattering is located within the matter-dominated phase. Thus, the time of last scattering,

$$
t_{ls} = 2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0(1+Z_{ls})}} - 2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}},
\tag{2.5}
$$

is given by a formula identical in form to (2.4) for t_{eq}, and satisfies $t_{eq} < t_{ls} < 0$.

B. The Radiation-Dominated Phase

Preceding the matter-dominated phase of expansion is a radiation-dominated phase of expansion. During this phase the scale factor is

$$
a(t) = C\sin(t+D) \qquad \text{for} \quad t_{end} < t < t_{eq}.
\tag{2.6}
$$

Here the constants C, D, t_{end} are defined by

$$
\begin{aligned}
C &= \frac{1}{H_0}\Omega_0^{1/2}(\Omega_0-1)^{-1}(1+Z_{eq})^{-1/2},\\
D &= 2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}} - \arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0(1+Z_{eq})}}, \quad \text{and} \\
t_{end} &= -2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}} + \arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0(1+Z_{eq})}} + \arcsin\sqrt{\frac{(\Omega_0-1)(1+Z_{eq})}{\Omega_0(1+Z_{end})^2}}.
\end{aligned}
\tag{2.7}
$$

During this radiation-dominated phase of expansion the energy density is proportional to $a^{-4}(t)$ and the pressure is equal to 1/3 of the energy-density. This phase is preceded by a de Sitter phase.

C. The Initial de Sitter (Inflationary) Phase

In our coordinate system, the de Sitter (exponentially expanding, inflationary) phase has scale factor

$$
a(t) = \frac{E}{\sin(t+F)} \qquad \text{for} \quad t < t_{end}.
\tag{2.8}
$$

Here the constants E and F are defined by

$$E = \frac{-1}{H_0}\Omega_0^{-1/2}(1+Z_{eq})^{1/2}(1+Z_{end})^{-2}, \quad \text{and} \tag{2.9}$$

$$F = 2\arcsin\sqrt{\frac{\Omega_0 - 1}{\Omega_0}} - \arcsin\sqrt{\frac{\Omega_0 - 1}{\Omega_0(1+Z_{eq})}} - 2\arcsin\sqrt{\frac{(\Omega_0-1)(1+Z_{eq})}{\Omega_0(1+Z_{end})^2}}.$$

Note that the constant $E < 0$ because $\sin(t+F) < 0$ during the de Sitter phase. During the de Sitter phase, the energy density is a constant

$$\rho_{\mathrm{deSitter}} = \frac{3}{8\pi}E^{-2} = \rho_0\frac{(1+Z_{end})^4}{1+Z_{eq}} = \frac{3H_0^2\Omega_0}{8\pi G}\frac{(1+Z_{end})^4}{1+Z_{eq}} \tag{2.10}$$

and the (negative) pressure is $-\rho_{\mathrm{deSitter}}$.

D. Properties of the Cosmological Model

It may be easily verified that the scale factor and its derivative w.r.t. time t are both continuous, however the second derivative is discontinuous. This is because in our simple inflationary model, the energy-density is a continuous function but the pressure changes discontinuously at the beginning and end of the radiation-dominated epoch.

The de Sitter phase "begins" at early times when the time coordinate t approaches the value $-\pi - F$. At this early time the cosmological scale factor is very large (approaching infinity as $t \to -\pi - F$). As the time coordinate increases, the scale factor decreases, eventually reaching a minimum value when $t = t_{min} = -\pi/2 - F$. After this time, the scale factor begins to increases again (exponentially in physical time). One might find it reasonable to demand that the universe be expanding at time t_{end} when the inflationary phase ends. This is the case if and only if $t_{min} < t_{end}$, which implies that the free parameters given in Table I must satisfy the inequality

$$\sqrt{\frac{(\Omega_0-1)(1+Z_{eq})}{\Omega_0}} < 1 + Z_{end}. \tag{2.11}$$

It is also easy to determine the "amount" of inflation that takes place. The amount that the universe has expanded between time t_{min}, when the spatial sections have their smallest extent, and time t_{end}, when the inflationary phase terminates and the radiation-dominated phase begins, is

$$\frac{a(t_{end})}{a(t_{min})} = (1+Z_{end})\sqrt{\frac{\Omega_0}{(\Omega_0-1)(1+Z_{eq})}}. \tag{2.12}$$

Comparison with (2.11) shows the obvious - if the universe is expanding at the end of the de Sitter phase, then the amount of inflationary expansion (2.12) is greater than unity. In typical inflationary models, the free parameters have values of order H_0 between 50 and 100 Km/s-Mpc, $\Omega_0 < 2$, $100 < Z_{ls} < 1500$, $2 \times 10^3 < Z_{eq} < 2 \times 10^4$, and $10^{26} < Z_{end}$.

There is a sense in which the spatially-closed inflationary models are not "natural." One of the principal motivations which led to the development of the inflationary paradigm was the desire to solve the so-called "horizon problem." As we will now show, this problem is only solved (for reasonable choices of the cosmological parameters) if $\Omega_0 < 2$. Thus, while it is technically consistent to use the results obtained in this paper for any value of $\Omega_0 > 1$, one must bear in mind that the cosmological model, for large values of Ω_0, runs counter to the spirit of inflation.

The horizon problem may be stated in terms of a set of points C, which is the intersection of the past horizon of an observer today with the surface of last scattering. The horizon problem is "solved" if C lies within the causal domain of influence of either (1) a point on the initial singularity, in a big bang model, or (2) a point on the surface at $t = t_{\rm min}$ where inflation "begins", in a model with no initial singularity. Thus, in our model, which is of type (2), the horizon problem is solved if and only if

$$|t_0 - t_{\rm ls}| < |t_{\rm ls} - t_{\rm min}| \tag{2.13}$$
$$\Longleftrightarrow$$
$$2\arcsin\sqrt{\tfrac{\Omega_0-1}{\Omega_0}} - 4\arcsin\sqrt{\tfrac{\Omega_0-1}{\Omega_0(1+Z_{\rm ls})}} + \arcsin\sqrt{\tfrac{\Omega_0-1}{\Omega_0(1+Z_{\rm eq})}} + 2\arcsin\sqrt{\tfrac{(\Omega_0-1)(1+Z_{\rm eq})}{\Omega_0(1+Z_{\rm end})^2}} < \tfrac{\pi}{2}.$$

For reasonable cosmological models, the terms containing $Z_{\rm ls}$, $Z_{\rm eq}$, and $Z_{\rm end}$ may be neglected. The horizon problem is then solved if and only if

$$\arcsin\sqrt{\frac{\Omega_0 - 1}{\Omega_0}} < \frac{\pi}{4} \iff \Omega_0 < 2. \tag{2.14}$$

While we present results for any value of Ω_0, the cosmological model itself should be viewed with some suspicion if Ω_0 is much larger than unity.

The final result of this paper are values of the dimensionless quantities

$$M_l \equiv \frac{\rho_{\rm Planck}}{\rho_{\rm deSitter}} \frac{l(l+1)}{6} \langle a_l^2 \rangle. \tag{2.15}$$

Here $\rho_{\rm Planck}$ is the Planck energy-density $\rho_{\rm Planck} = \frac{1}{\hbar G^2} \approx 5 \times 10^{93}\,{\rm gm/cm^3}$. It will turn out that M_l is independent of H_0, and depends only upon the dimensionless quantities Ω_0, $Z_{\rm ls}$, $Z_{\rm eq}$, and $Z_{\rm end}$. This is because $\rho_{\rm deSitter}$, and $\langle a_l^2 \rangle$ are both proportional to H_0^2. In addition, if $Z_{\rm end}$ is sufficiently large then the M_l are also independent of its value.

E. The Sachs-Wolfe Effect

If the metric perturbation h_{ij} vanishes and the temperature of the CBR on the surface of last scattering is constant, an observer today would see exactly the same temperature at each point on the celestial sphere, and $C(\gamma)$ would vanish. However the metric perturbations will in general break the rotational symmetry and perturb the energy of the photons. This results in a temperature fluctuation which varies from point to point on the celestial sphere; the fluctuation may be calculated in the same way as for a spatially-flat Universe, given in [5].

We assume that the observer is located at $t = 0$, and at "radial" coordinate $\chi = 0$. (Because the coordinate system is singular at $\chi = 0$ every value of θ, ϕ corresponds to the same space-time point at $\chi = 0$, so their values are irrelevant when $\chi = 0$.) If the observer looks out at a point Ω on the celestial sphere, she observes photons that arrive, in the unperturbed metric, along the null geodesic path

$$t(\lambda) = \lambda \quad \chi(\lambda) = |\lambda| = -\lambda \quad \theta(\lambda) = \theta_\Omega \quad \phi(\lambda) = \phi_\Omega. \tag{2.16}$$

In these equations, θ_Ω and ϕ_Ω are the angular coordinates of the point Ω on the celestial two-sphere. We have chosen the (non-affine) parameter λ along the null geodesic path to run through the range $t_{ls} \le \lambda \le 0$ between the time of last scattering and the observation today.

In the presence of the metric perturbation h_{ij} the fractional temperature fluctuation observed at point Ω on the celestial sphere is

$$\frac{\delta T}{T}(\Omega) = -\frac{1}{2}\int_{t_{ls}}^{0} d\lambda \left(\frac{\partial h_{xx}}{\partial t}\right)(t(\lambda), \chi(\lambda), \theta(\lambda), \phi(\lambda)). \tag{2.17}$$

As indicated in this formula, the partial derivative w.r.t. the time coordinate t is taken before setting the coordinates equal to the values which they take along the unperturbed null geodesic path.

III. THE METRIC PERTURBATION OPERATOR

The classical metric perturbation h_{ij} may be replaced with a quantum field operator. The justification for this is given in our detailed paper on the spatially-flat case [3] and will not be repeated here. The basic idea is that the inflationary epoch redshifts away all the perturbations, with the exception of the zero-point quantum fluctuations. Hence we calculate the expectation value of $\frac{\delta T}{T}(\Omega)\frac{\delta T}{T}(\Omega')$ in the vacuum state $|0\rangle$ appropriate to the initial de Sitter state.

A. Mode function expansion of Metric Perturbation Operator

The quantum field operator (which we denote with the same symbol h_{ij} as the corresponding classical perturbation) may be expanded in terms of a complete set of mode functions. As was originally shown by Ford and Parker [6], in an FRW cosmological model, the time-dependence of these mode functions is the same as that of a massless minimally-coupled scalar field. The field operator is

$$\begin{aligned} h_{ij}(t,\chi,\theta,\phi) = & \\ \sum_{L=2}^{\infty}\sum_{l=2}^{L}\sum_{m=-l}^{l}\Big[& \psi_{Llm}(t)T_{ij}^{(s;Llm)}(\chi,\theta,\phi)c_{Llm} + \psi^*_{Llm}(t)T_{ij}^{*(s;Llm)}(\chi,\theta,\phi)c^\dagger_{Llm} \\ & + \psi_{Llm}(t)T_{ij}^{(v;Llm)}(\chi,\theta,\phi)d_{Llm} + \psi^*_{Llm}(t)T_{ij}^{*(v;Llm)}(\chi,\theta,\phi)d^\dagger_{Llm}\Big]. \end{aligned} \tag{3.1}$$

In this expression, the sum is over a complete set of rank-two symmetric transverse traceless tensors $T_{ij}^{(Llm)}$. These tensor modes are defined on a unit-radius sphere S^3 and are given explicitly by Higuchi [7]. (Note however a typo [8] in one of the formulae which does not affect the results which we need.) Henceforth we will denote the triple sum that appears in

(3.1) by $\sum_{Llm}$ without explicitly indicating the ranges of summation. (The temperature T can always be distinguished from the tensor modes $T_{ij}^{(Llm)}$, since the latter is always written with indices.)

The graviton has two possible polarization states, labeled s and v in this expansion, each of which has its own set of tensor modes. The modes are labeled by the three integers L, l, and m. (Note that Ford and Parker's [6] index $n = L + 1$ in our notation.) Associated with the s-polarization modes are creation and annihilation operators c_{Llm} and $c_{Llm}^{\dagger}$, and associated with the v-polarization modes are creation and annihilation operators d_{Llm} and $d_{Llm}^{\dagger}$. The only non-vanishing commutation relation among this infinite set of operators is the relation

$$[c_{Llm}, c^{\dagger}_{L'l'm'}] = [d_{Llm}, d^{\dagger}_{L'l'm'}] = \delta_{LL'}\delta_{ll'}\delta_{mm'} \tag{3.2}$$

where δ denotes the Kronecker delta function.

B. The Transverse-Traceless-Symmetric Tensor Harmonics

The tensor modes defined by Higuchi [7] obey the normalization condition

$$\int_0^{\pi} d\chi \sin^2\chi \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi T_{ij}^{(p;Llm)} T_{rs}^{(p';L'l'm')} P^{ir} P^{js} = \delta_{LL'}\delta_{ll'}\delta_{mm'}\delta_{pp'}. \tag{3.3}$$

Here, the polarization indices p and p' take on either of the values "s" or "v". The integral is over the unit-radius three-sphere, and P^{ir} is the inverse of the metric on the unit-radius three-sphere:

$$P_{ij}dx^i dx^j = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4}$$

Note that the measure that appears in the normalization integral (3.3) is the usual volume element defined by $\sqrt{\det P_{ij}}$. The Sachs-Wolfe effect (2.17) is produced only by the $\chi\chi$ component of h_{ij}. Because $T_{\chi\chi}^{(v;Llm)} \equiv 0$, only the "s" polarization state contributes to the temperature fluctuation. The only component needed is thus

$$T_{\chi\chi}^{(s;Llm)}(\chi,\theta,\phi) \equiv R_L^l(\chi) Y_{lm}(\theta,\phi). \tag{3.5}$$

The $Y_{lm}(\theta,\phi)$ are standard scalar spherical harmonic functions on the two-sphere [9]. The "radial" dependence is given by

$$R_L^l(\chi) \equiv \sqrt{\frac{(l-1)l(l+1)(l+2)(L+1)(L+l+1)!}{2L(L+1)^2(L+2)(L-l)!}} (\sin\chi)^{-5/2}\, \mathrm{P}_{L+1/2}^{-(l+1/2)}(\cos\chi), \tag{3.6}$$

where the functions $\mathrm{P}_{L+1/2}^{-(l+1/2)}(z)$ are associated Legendre functions [9]. In Section V we explain how these functions may be easily evaluated.

C. Normalization Condition for Wavefunctions

The quantum field operator h_{ij} obeys canonical commutation relations which can be derived from the quadratic part of the gravitational action. We have already specified the normalization of the creation/annihilation operators (3.2) and of the spatial part of the mode functions (3.3). The commutation relations for h_{ij} then determines the normalization of the time part $\psi_{Llm}(t)$ of the graviton wavefunctions. The details of this procedure are given in [6] and yield a normalization condition

$$\psi_{Llm}(t)\frac{d}{dt}\psi^*_{Llm}(t) - \psi^*_{Llm}(t)\frac{d}{dt}\psi_{Llm}(t) = 32i\pi\hbar G a^{-2}(t). \tag{3.7}$$

(Note that the normalization condition given in equation (3.3) of Ford and Parker [6] contains a minor typo [10].)

D. Choice Of An Initial (Vacuum) State

If one defines a Fock vacuum state by the property that it is annihilated by all of the operators c_{Llm} and d_{Llm}, then the choice of vacuum state is really determined by the choice of the mode functions $\psi_{Llm}(t)$. For the reasons given in [3] we choose these mode functions to be those which correspond to the unique de Sitter invariant vacuum state $|0\rangle$ during the initial inflationary stage whose two-point function has Hadamard form.

E. Wavefunction During The De Sitter Phase

As shown in equation (2.18) of Ford and Parker [6] the mode functions obey the minimally-coupled massless scalar wave equation

$$\left[\frac{d^2}{dt^2} + \frac{2}{a}\frac{da}{dt}\frac{d}{dt} + L(L+2)\right]\psi_{Llm}(t) = 0. \tag{3.8}$$

There is a slight subtlety: it is impossible to define a de Sitter invariant Fock vacuum state for the minimally-coupled massless scalar field [11]. However it was shown by Allen and Folacci [12] that the difficulty only arises for the $L = 0$ mode. In the case of the gravitational field operator, the $L = 0$ and the $L = 1$ modes are both absent; they correspond to "monopole" and "dipole" dynamical degrees of freedom which are not present in the spin-two case. Hence in the case of the gravitational field, it is possible to define the desired de Sitter invariant vacuum state. The corresponding normalized wavefunction during the de Sitter phase is [2,7,11–14]

$$\psi_{Llm}(t) = \psi_L(t) = \frac{1}{a}\sqrt{\frac{16\pi\hbar G}{L(L+1)(L+2)}}\left(i(L+1) - \frac{1}{a}\frac{da}{dt}\right)e^{-i(L+1)t} \quad \text{for} \quad t < t_{\rm end}. \tag{3.9}$$

We note in passing that this time-dependent part of the wavefunction depends only upon L and not upon l and m. (This guarantees that the vacuum state will be invariant under all rotations of the three-sphere $t =$ constant, which is the subgroup $SO(4)$ of the de Sitter group $SO(1,4)_0$. However the invariance of the state under the de Sitter group $SO(1,4)_0$ is *not* obvious from inspection. Here the subscript on $SO(1,4)$ denotes the part of the group connected to the identity.) For this reason, from this point on we drop the indices l, m from the time-dependent part of the wavefunction, denoting ψ_{Llm} by ψ_L. Because the wave equation (3.8) is a second-order ODE, the solution $\psi_L(t)$ during the de Sitter phase completely determines the solution at all later times. The solution at later times is conveniently written in terms of Bogoliubov coefficients.

F. Wavefunction During The Radiation-Dominated Phase

The epoch that follows the de Sitter epoch is the radiation-dominated phase. One may write the solution to the wave equation during this phase as

$$\psi_L(t) = \alpha_L^{rad}\psi_L^{rad}(t) + \beta_L^{rad}\psi_L^{*rad}(t) \quad \text{for} \quad t_{end} < t < t_{eq}. \tag{3.10}$$

Here, the positive frequency mode during the radiation epoch is defined by

$$\psi_L^{rad}(t) = \frac{1}{a}\sqrt{\frac{16\pi\hbar G}{(L+1)}}e^{-i(L+1)t} \quad \text{for} \quad t_{end} < t < t_{eq}. \tag{3.11}$$

The Bogoliubov coefficients are determined by a condition which follows from the wave equation (3.8): both $\psi_L(t)$ and its time derivative must be continuous at all times, and in particular at $t = t_{end}$. One obtains

$$\alpha_L^{rad} = (L(L+2))^{-1/2}\Big(i(L+1) + \sqrt{Q-1} - \frac{iQ}{2(L+1)}\Big) \tag{3.12}$$
$$\beta_L^{rad} = \frac{i}{2}(L+1)^{-1}(L(L+2))^{-1/2}Qe^{-2i(L+1)t_{end}}.$$

Here Q is the constant defined by

$$Q = \frac{\Omega_0(1+Z_{end})^2}{(\Omega_0 - 1)(1+Z_{eq})}. \tag{3.13}$$

We stress once again that *the solution $\psi_L(t)$ during the de Sitter phase completely determines the solution at all later times.* In other words *the choice of a "positive-frequency" mode function during the radiation phase is unimportant.* Had we picked a different solution to the wave equation (3.8) to call "positive frequency" then α_L^{rad} and β_L^{rad} would have changed in such a way as to keep the mode function $\psi_L(t)$ given in (3.10) unchanged. In similar fashion, the solution of the wave equation during the radiation phase completely determines its solution during the matter-dominated phase.

G. Wavefunction During The Matter-Dominated Phase

The wavefunction during the matter-dominated (dust) phase may again be expressed as a linear combination of the natural positive-frequency solution and its complex conjugate:

$$\psi_L(t) = \alpha_L \psi_L^{mat}(t) + \beta_L \psi_L^{*mat} \quad \text{for} \quad t_{eq} < t < 0. \tag{3.14}$$

The positive frequency mode functions during the matter epoch are

$$\psi_L^{mat}(t) = \frac{1}{a}\sqrt{\frac{16\pi\hbar G}{(L+1)(2L+1)(2L+3)}}\left(2i(L+1) + \frac{1}{a}\frac{da}{dt}\right)e^{-i(L+1)t} \quad \text{for} \quad t_{eq} < t < 0. \tag{3.15}$$

The Bogoliubov coefficients α_L and β_L are determined (as in the spatially flat case [3]) by combining the Bogoliubov coefficients for the two different phases.

$$\begin{pmatrix} \alpha_L & \beta_L \\ \beta_L^* & \alpha_L^* \end{pmatrix} = \begin{pmatrix} \alpha_L & \beta_L \\ \beta_L^* & \alpha_L^* \end{pmatrix}^{rad} \begin{pmatrix} \alpha_L & \beta_L \\ \beta_L^* & \alpha_L^* \end{pmatrix}^{mat}. \tag{3.16}$$

As previously, the Bogoliubov coefficients α_L^{mat} and β_L^{mat} are determined by matching the positive frequency radiation mode function $\psi_L^{rad}(t)$ to the linear combination $\alpha_L^{mat}\psi_L^{mat}(t) + \beta_L^{mat}\psi_L^{*mat}$ at time t_{eq}. One obtains

$$\begin{aligned} \alpha_L^{mat} &= ((2L+1)(2L+3))^{-1/2}\left(-2i(L+1) + \sqrt{W-1} + \frac{iW}{4(L+1)}\right) \\ \beta_L^{mat} &= \frac{-i}{4}(L+1)^{-1}((2L+1)(2L+3))^{-1/2} W e^{-2i(L+1)t_{eq}}. \end{aligned} \tag{3.17}$$

Here the constant W is given by

$$W = \frac{\Omega_0(1+Z_{eq})}{\Omega_0 - 1}. \tag{3.18}$$

We are now in a position to evaluate the multipole moments $\langle a_l^2 \rangle$ of the angular correlation function $C(\gamma)$.

IV. MULTIPOLE MOMENTS OF $C(\gamma)$

Combining the results of the previous section, one can easily obtain a formula for the multipole moments of the angular correlation function $C(\gamma)$. One replaces the metric perturbation that appears in the Sachs-Wolfe formula (2.17) with expansion (3.1) of the field operator. The resulting operator depends upon an angle Ω on the celestial sphere. One then takes the expectation value of this operator with an identical operator at a different point Ω' on the celestial sphere. This yields the correlation function

$$C(\gamma) \equiv \left\langle 0 \middle| \tfrac{\delta T}{T}(\Omega)\tfrac{\delta T}{T}(\Omega') \middle| 0 \right\rangle = \tag{4.1}$$

$$\tfrac{1}{4}\int_{t_{ls}}^0 dt \int_{t_{ls}}^0 dt' \sum_{Llm}\sum_{L'l'm'} \dot{\psi}_L(t)\dot{\psi}_L^*(t') R_L^l(|t|) R_{L'}^{l'}(|t'|) Y_{lm}(\Omega) Y_{l'm'}^*(\Omega') \left\langle 0 \middle| c_{Llm} c_{L'l'm'}^\dagger \middle| 0 \right\rangle.$$

Here γ is the angle between the points Ω and Ω' on the celestial sphere. Because the Sachs-Wolfe formula (2.17) involves the time-derivative of the mode function, we have defined

$\dot{\psi}_L(t) \equiv d\psi_L(t)/dt$, where ψ_L is the mode function during the matter-dominated epoch, given in (3.14).

To simplify this expression, first note that the matrix element $\left\langle 0 \middle| c_{Llm} c^\dagger_{L'l'm'} \middle| 0 \right\rangle = \delta_{LL'}\delta_{ll'}\delta_{mm'}$. This eliminates the triple sum $\sum_{L'l'm'}$. Because the summand is independent of the summation index m, one may then explicitly carry out the sum over m using the addition formula for spherical harmonics, equation (3.62) of reference [15]:

$$\sum_{m=-l}^{l} Y_{lm}(\Omega) Y^*_{l'm'}(\Omega') = \frac{2l+1}{4\pi} P_l(\cos\gamma). \tag{4.2}$$

Because the argument of the Legendre function $P_l(x)$ is the cosine of the angle γ between the points on the celestial sphere, this shows explicitly that the correlation function depends only upon γ.

$$C(\gamma) = \frac{1}{4} \sum_{L=2}^{\infty} \sum_{l=2}^{L} \frac{2l+1}{4\pi} P_l(\cos\gamma) \int_{t_{ls}}^{0} dt \int_{t_{ls}}^{0} dt' \dot{\psi}_L(t) \dot{\psi}^*_L(t') R^l_L(|t|) R^l_L(|t'|) \tag{4.3}$$

Comparing this to the definition of the multipole moments (1.1), and noting that the summation $\sum_{L=2}^{\infty}\sum_{l=2}^{L}$ is equivalent to the summation $\sum_{l=2}^{\infty}\sum_{L=l}^{\infty}$, one immediately obtains a simple formula for the multipole moment,

$$\langle a_l^2 \rangle = \frac{1}{4} \sum_{L=l}^{\infty} \int_{t_{ls}}^{0} dt \int_{t_{ls}}^{0} dt' \dot{\psi}_L(t) \dot{\psi}^*_L(t') R^l_L(|t|) R^l_L(|t'|) = \frac{1}{4} \sum_{L=l}^{\infty} |\alpha_L I^l_L + \beta_L I^{l\,*}_L|^2. \tag{4.4}$$

The complex quantity I^l_L is what remains of the integral of the mode function along the radial null geodesic path.

$$I^l_L \equiv \int_{t_{ls}}^{0} dt R^l_L(|t|) \frac{d}{dt} \psi^{mat*}_L(t) \tag{4.5}$$

Note that we have assumed (as is implied in Table I) that the surface of last scattering lies within the matter-dominated epoch; the positive frequency mode function during the matter phase is given by (3.15). The Bolgoliubov coefficients are given by (3.16)

$$\alpha_L = \alpha_L^{rad} \alpha_L^{mat} + \beta_L^{rad} \beta_L^{*mat} \quad \text{and} \quad \beta_L = \alpha_L^{rad} \beta_L^{mat} + \beta_L^{rad} \alpha_L^{*mat}, \tag{4.6}$$

where the Bolgoliubov coefficients for the matter and radiation transitions are defined by (3.17) and (3.12). In the next section, we discuss how the multipole moments $\langle a_l^2 \rangle$ may be rapidly evaluated using numerical techniques.

V. NUMERICAL EVALUATION OF THE MULTIPOLE MOMENTS

As discussed at the end of Section II, it is convenient to define dimensionless quantities $M_l \equiv \frac{\rho_{Planck}}{\rho_{deSitter}} \frac{l(l+1)}{6} \langle a_l^2 \rangle$. Using the previous formulae one may write this in the dimensionless form

$$M_l = \frac{32\pi^2}{3}\frac{l(l+1)}{6}\frac{1+Z_{eq}}{(1+Z_{end})^4}\left(\frac{\Omega_0-1}{\Omega_0}\right)^3 \sum_{L=l}^{\infty} \frac{1}{L(L+1)(L+2)}|\alpha_L J_L^l + \beta_L J_L^{l*}|^2. \quad (5.1)$$

where

$$J_L^l \equiv \int_{t_{ls}}^0 dt R_L^l(|t|) \csc^2(t/2+B) e^{-i(L+1)t} \times \quad (5.2)$$
$$\left[-3i(L+1)\cot(t/2+B) - \tfrac{3}{2}\csc^2(t/2+B) + 2L^2 + 4L + 1)\right].$$

Taken together with the definitions of α_L and β_L given (4.6), (3.17), and (3.12), the constants Q and W defined in (3.13) and (3.18), and the radial function $R_L^l(\chi)$ defined in (3.6) this is a self-contained formula for calculating M_l.

Before discussing the evaluation of M_l in general, it is worth commenting on two limits. The first limit is the $Z_{end} \to \infty$ case, where the amount of inflation is large. In this case, it is easy to see that Q and hence α_L and β_L diverge $\propto (1+Z_{end})^2$. Thus in the limit, M_l converges. A second interesting limit is the spatially-flat one, $\Omega_0 - 1 \to 0^+$, where the density parameter approaches unity from above. In this case, it is easy to see that Q and hence α_L and β_L diverge as $(\Omega_0-1)^{-1}$, and the integral J_L^l diverges as $(\Omega_0-1)^{-1/2}$. Once again, the limit is well-defined. In addition, in this case, the sum over L can be re-written as an integral, recovering the $k=0$ spatially-flat formula given in [3].

We evaluated M_l using an fourth-order Runge-Kutta adaptive stepsize integrator [16] to obtain the integral which defines J_L. In cases of interest, one frequently needs to include many values of L in the summation. In practice we found that summing over the range $L \in l, l+1, \cdots, l_{max}$ with $l_{max} = 32 + (5l+10)/|t_{ls}|$ gave results accurate to a few percent for reasonable ranges of the free parameters listed in Table I. Rather than compute the J_L^l one at a time, it is more practical to compute them "en masse", determining $J_l^l, J_{l+1}^l, \cdots, J_{l_{max}}^l$ simultaneously. This can be done easily because the associated Legendre functions may be computed with a stable upwards recursion relation.

A. Second-Order Recursion Relations

The upwards recursion relation for the associated Legendre functions is given in equation (8.731.2) of reference [17].

$$P_{j+l+1/2}^{-(l+1/2)}(z) = \frac{2(l+j)}{2l+j+1} z P_{(j-1)+l+1/2}^{-(l+1/2)}(z) + \frac{1-j}{2l+j+1} P_{(j-2)+l+1/2}^{-(l+1/2)}(z) \quad \text{for} \quad j = 2, 3, \cdots, \quad (5.3)$$

together with the boundary conditions (or initial values) given in equation (8.755.1) of reference [17]:

$$P_{l+1/2}^{-(l+1/2)}(\cos\chi) = \frac{1}{\Gamma(l+3/2)}\left(\frac{\sin\chi}{2}\right)^{l+1/2} \quad \text{and} \quad P_{l+3/2}^{-(l+1/2)}(z) = z P_{l+1/2}^{-(l+1/2)}(z). \quad (5.4)$$

These relations may be used to obtain a recursion relation and initial values for the radial functions R_L^l. The initial values are

$$R_l^l(\chi) = \sqrt{\frac{(l-1)\Gamma(l+1)}{2\sqrt{\pi}\Gamma(l+3/2)}}(\sin\chi)^{l/2-1} \quad \text{and} \quad R_{l+1}^l(\chi) = \cos\chi\sqrt{\frac{2l(l+1)}{l+3}}R_l^l(\chi) \tag{5.5}$$

and the recursion relation is obtained from (5.3)

$$\text{for} \quad j = 2,3,\cdots \quad R_{l+j}^l(\chi) = \tag{5.6}$$

$$\sqrt{\frac{4(l+j)^2(l+j-1)}{j(l+j+2)(2l+j+1)}}\cos\chi R_{l+j-1}^l(\chi) - \sqrt{\frac{(j-1)(2l+j)(l+j-1)(l+j-2)}{j(2l+j+1)(l+j+1)(l+j+2)}}R_{l+j-2}^l(\chi)$$

Although this recursion relation does not appear to be stable, our experience has been that it accurately determines R_L^l for $l \le L \le l + 6000$.

VI. NUMERICAL RESULTS AND CONCLUSIONS

The numerical results are presented as a series of graphs Figs. 1, 2, and 3 showing the values of $M_l \equiv \frac{\rho_{\rm Planck}}{\rho_{\rm deSitter}}\frac{l(l+1)}{6}\langle a_l^2\rangle$. For all of these graphs, we have taken $Z_{\rm end} = 10^{26}$, $Z_{\rm eq} = 10^4$ and $Z_{\rm ls} = 1300$, and varied the density parameter Ω_0. The graphs also show the values of M_l for the spatially-flat $k = 0$ case, taken from [3]. This case corresponds to the critically-bound $\Omega_0 \to 1$ limit.

It is clear from the figures that this limit is quickly approached; when $\Omega_0 = 1.1$ the M_l are almost indistinguishable from the $k = 0$ spatially-flat case. It is not hard to see why. The effects of the spatial curvature only appear if the past light cone of the observer, taken back to the surface of last scattering, actually "probes" a substantial fraction of the spatial three-sphere. If the past light cone fails to do this, then within the past light cone the universe is indistinguishable (to good approximation) from a spatially-flat model.

The fraction of the three-sphere (S^3) within this past light cone is easy to determine. The three-volume contained within angle $\chi_{\rm max}$ from the point $\chi = 0$ of the unit-radius S^3 may be obtained by integrating $\sqrt{\det P_{ij}}$ where P_{ij} is the three-metric (3.4). One obtains $V(\chi_{\rm max}) = \pi(2\chi_{\rm max} - \sin 2\chi_{\rm max})$. The total volume of S^3 is $V(\pi) = 2\pi^2$. If we assume that $Z_{\rm ls}$ is much larger than one, then the fraction of the volume of S^3 contained within the past light cone is approximately

$$f(\Omega_0) \equiv \frac{V(|t_{\rm ls}|)}{V(\pi)} \approx \frac{V(2\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}})}{2\pi^2}. \tag{6.1}$$

For Ω_0 near 1, this fraction is well-appoximated by

$$f(\Omega_0) \approx \frac{16}{3\pi}(\Omega_0 - 1)^3. \tag{6.2}$$

Thus when $\Omega_0 = 1.1$ the past light cone only explores about 1/1000 of the spatial volume. Even if $\Omega_0 = 2$ the fraction of the three-sphere that is observed is only $f(\Omega_0 = 2) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \approx 0.1955\cdots$. This is why the multipole moments are not very sensitive to Ω_0 provided it is close to unity.

The extension of this calculation to the case of a spatially open FRW universe appears straightforward. However it turns out to be much more difficult than expected, primarily because the correct choice of initial state is not the obvious one, and because the final result for the multipole moments appears to contain logarithmic (infra-red) divergences at zero frequency. The spatially open case will be the subject of a forthcoming paper.

ACKNOWLEDGMENTS

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[17] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Corrected and Enlarged ed. (Academic, New York, 1980). This reference uses two symbols to refer to associated Legendre functions: P and P. However the definitions of these symbols given in the first two paragraphs of page 999 is incorrect. The first paragraph should refer to "straight" P and the second paragraph to "slant" P. The functions used in this paper are the "straight" P.

FIGURES

FIG. 1. The normalized multipole moments $M_l \equiv \frac{\rho_{\rm Planck}}{\rho_{\rm deSitter}} \frac{l(l+1)}{6} \langle a_l^2 \rangle$ of the CBR temperature fluctuations are shown as a function of the multipole number l, for a spatially-flat ($\Omega_0 = 1$) and for spatially-closed ($\Omega_0 > 1$) cosmological models. $\Omega_0 - 1$ needs to be fairly large for the effects of the spatial curvature to be significant. The models being compared all have cosmological parameters defined by the redshifts $Z_{\rm ls} = 1300$, $Z_{\rm eq} = 10^4$ and $Z_{\rm end} = 10^{26}$.

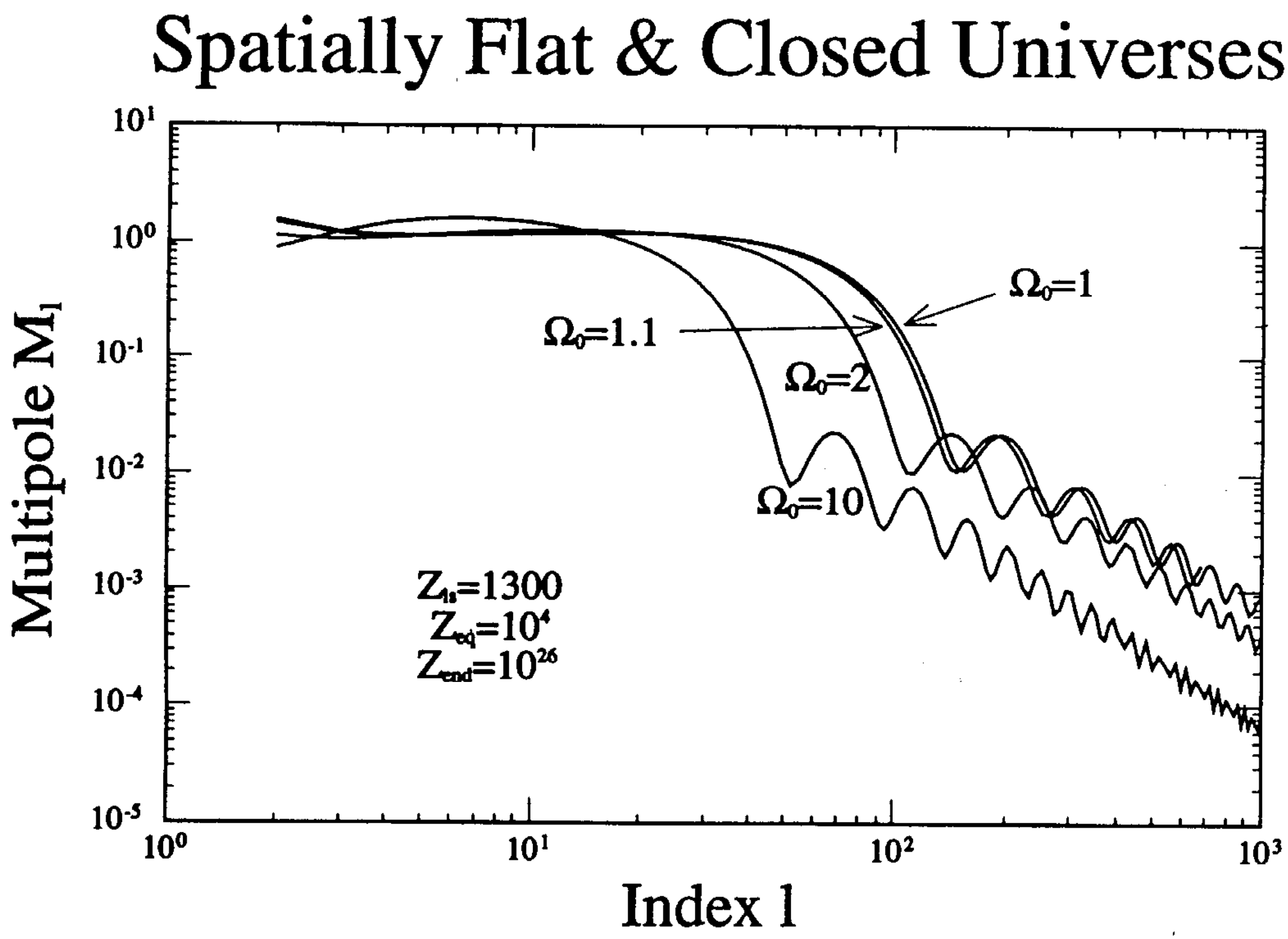
FIG. 2. The normalized multipole moments $M_l \equiv \frac{\rho_{\rm Planck}}{\rho_{\rm deSitter}} \frac{l(l+1)}{6} \langle a_l^2 \rangle$ are shown as a function of $\Omega_0 - 1$ for $l = 2, 5, 10, 20, 30, 50$. In all cases, the models being compared have the same cosmological parameters as in Fig. 1. Only when Ω_0 becomes significantly larger than one do the multipole moments change significantly from the spatially-flat case.

FIG. 3. The normalized multipole moments $M_l \equiv \frac{\rho_{\rm Planck}}{\rho_{\rm deSitter}} \frac{l(l+1)}{6} \langle a_l^2 \rangle$ are shown as a function of $\Omega_0 - 1$ for $l = 100, 200, 400$. In all cases, the models being compared have the same cosmological parameters as in Figs. 1 and 2. Only when Ω_0 becomes significantly larger than one do the multipole moments change significantly from the spatially-flat case.

TABLES

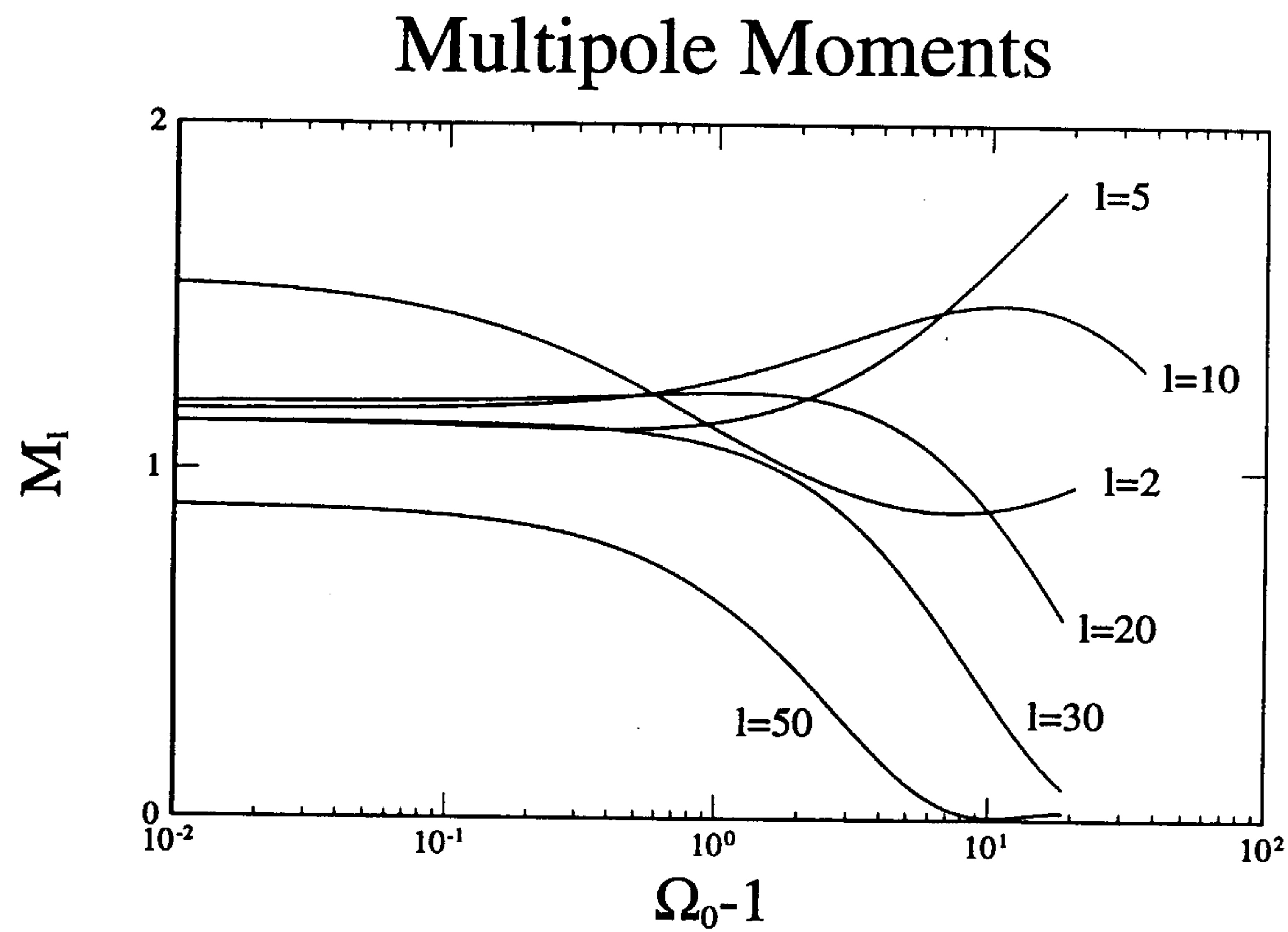
TABLE I. List of the free parameters that define the cosmological model.

Parameter	Units	Range	Description
H_0	length^{-1}	$H_0 > 0$	Present-day Hubble expansion rate
Ω_0	dimensionless	$\Omega_0 > 1$	Present-day density parameter
Z_{ls}	dimensionless	$Z_{ls} > 0$	Redshift at last scattering of CBR
Z_{eq}	dimensionless	$Z_{eq} > Z_{ls}$	Redshift at equal matter/radiation energy density
Z_{end}	dimensionless	$Z_{end} > Z_{eq}$	Redshift at end of de Sitter inflation



Spatially Flat & Closed Universes
Multipole M_l
Index l
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^0
10^1
10^2
10^3
Ω_0=1
Ω_0=1.1
Ω_0=2
Ω_0=10
Z_{ls}=1300
$Z_{eq}=10^4$
$Z_{end}=10^{26}$

FIGURE 1



Multipole Moments
M_l
2
1
0
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Ω_0-1
l=5
l=10
l=2
l=20
l=30
l=50

FIGURE 2

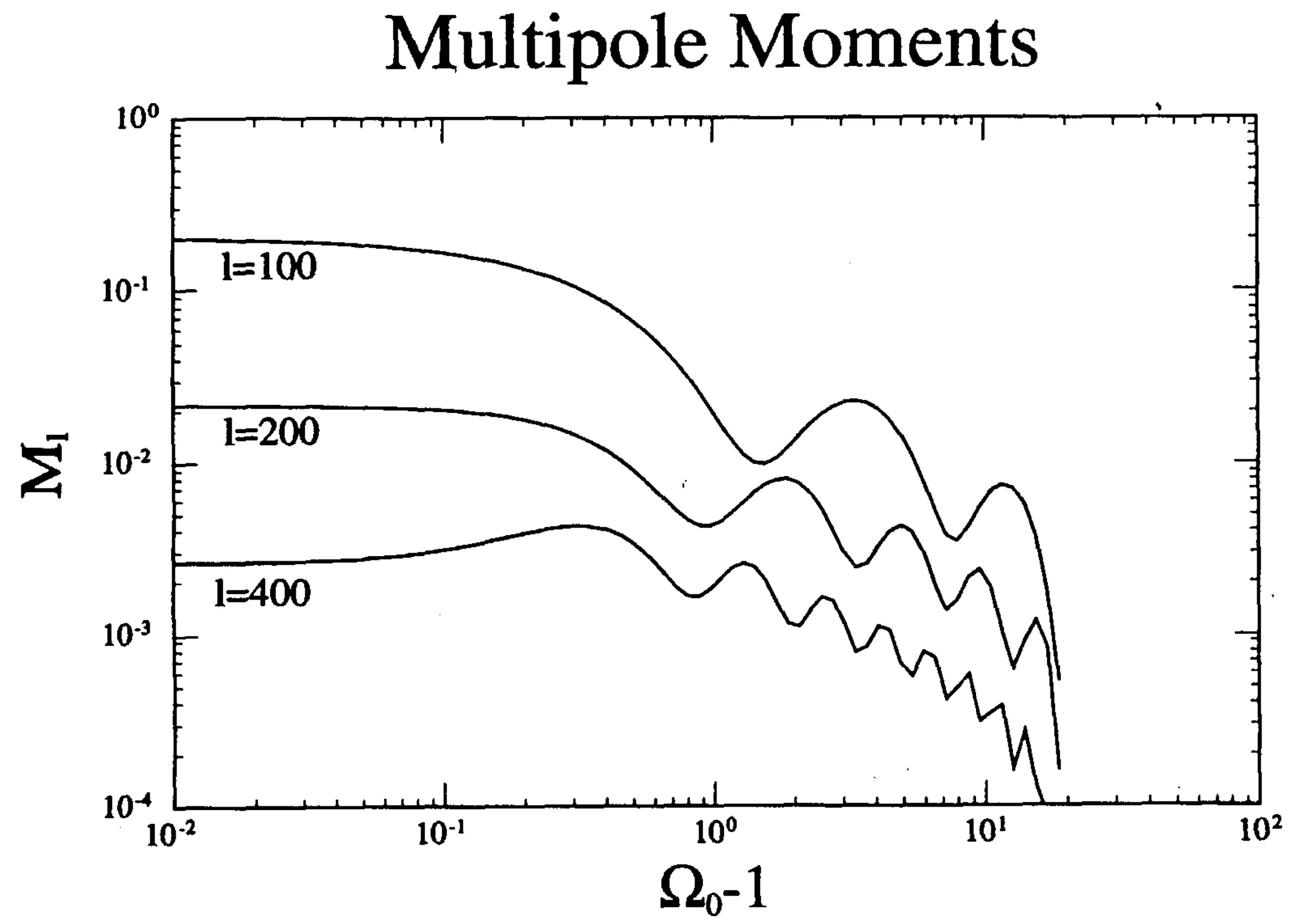

Multipole Moments
M_l
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
l=100
l=200
l=400
10^{-2}
10^{-1}
10^0
10^1
10^2
Ω_0-1

FIGURE 3